Topology optimization I: Trusses

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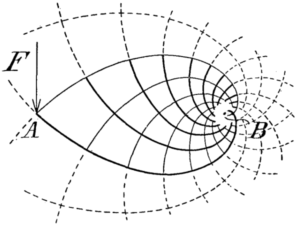
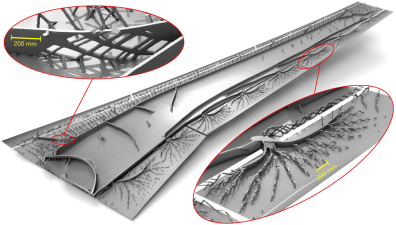
- Truss structures possess high stiffness-to-weight ratio
- Truss-like or lattice structures are often achieved by continuum topology optimization
- There are convex (and easily solvable) formulations
- Quite mature theory (pioneered by Michell in 1904)

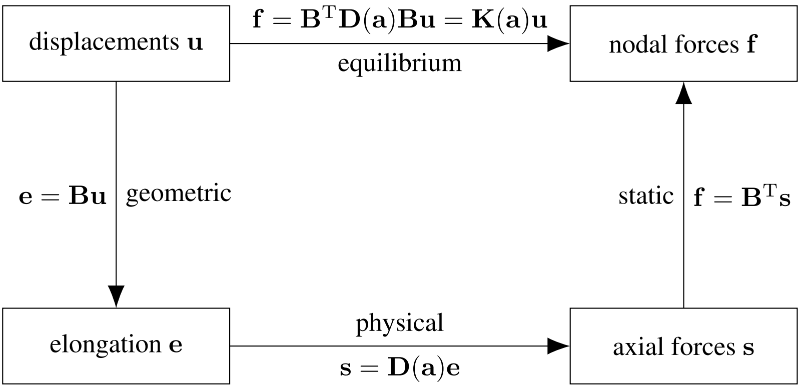


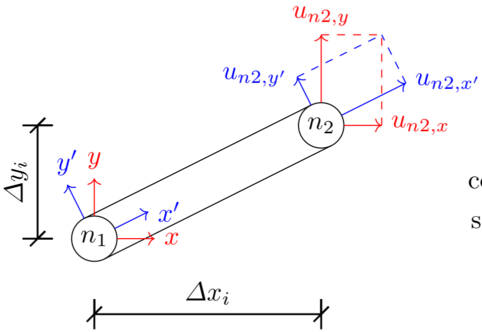
(a) Michell-type structure. Courtesy of A. G. M. Michell, The limits of economy of material in frame-structures, *Philosophical Magazine Series 6*, 8(47):589–597, 1904, doi: 10.1080/14786440409463229



(b) Lattice-like Wing structure. Courtesy of N. Aage, E. Andreassen, B. S. Lazarov, and O. Sigmund, Giga-voxel computational morphogenesis for structural design, *Nature*, 550(7674):84–86, 2017, doi: 10.1038/nature23911

Finite elements overview


displacements $\mathbf{u}$
$\mathbf{f} = \mathbf{B}^{\mathrm{T}}\mathbf{D}(\mathbf{a})\mathbf{B}\mathbf{u} = \mathbf{K}(\mathbf{a})\mathbf{u}$
equilibrium
nodal forces $\mathbf{f}$
$\mathbf{e} = \mathbf{B}\mathbf{u}$
geometric
static
$\mathbf{f} = \mathbf{B}^{\mathrm{T}}\mathbf{s}$
elongation $\mathbf{e}$
physical
$\mathbf{s} = \mathbf{D}(\mathbf{a})\mathbf{e}$
axial forces $\mathbf{s}$



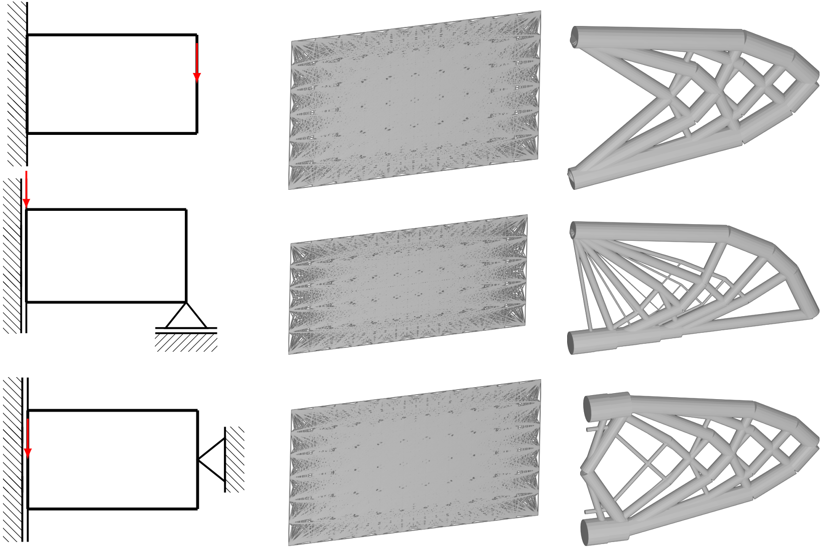
$$\ell_i = \|\begin{pmatrix}\Delta x & \Delta y\end{pmatrix}\|_2$$
$$\cos(\alpha_i) = \Delta x_i / \ell_i$$
$$\sin(\alpha_i) = \Delta y_i / \ell_i$$

$$\begin{aligned}
e_i &= (u_{n_{2,x}} - u_{n_{1,x}})\cos(\alpha_i) + (u_{n_{2,y}} - u_{n_{1,y}})\sin(\alpha_i) \\
&= \begin{pmatrix}-\cos(\alpha_i) & -\sin(\alpha_i) & \cos(\alpha_i) & \sin(\alpha_i)\end{pmatrix}\begin{pmatrix}u_{n_{1,x}} & u_{n_{1,y}} & u_{n_{2,x}} & u_{n_{2,y}}\end{pmatrix}^{\mathrm{T}} \\
&= \mathbf{b}_i \mathbf{u} \\
d_i &= E_i a_i / \ell_i \\
\mathbf{K}_i(a_i) &= \mathbf{b}_i^{\mathrm{T}} d_i(a_i) \mathbf{b}_i \qquad \longrightarrow \mathbf{K}_i(a_i) \in \mathbb{S}^4_{\succeq 0} \text{ and has rank } 1
\end{aligned}$$

Ground structure method

- Approximation of the continuum design domain Ω by discretization $\tilde{\Omega} := G(\mathcal{N}, \mathcal{E})$, where $\mathcal{N} = \{1, \dots, n_\text{n}\}$ is the set of nodes and $\mathcal{E} = \{\{n_{1,1}, n_{2,1}\}, \dots, \{n_{1,n_\text{e}}, n_{2,n_\text{e}}\}\}$ with $n_{\bullet,\bullet} \in \mathcal{N}$ standing for the set of elements (bars)
- Topology optimization (TO): which elements from $\mathcal{E}$ appear in an optimal design to
  - reach target/optimize performance
  - reach target/minimize weight
  - satisfy/optimize other constraints/objectives (resonance frequency, buckling, manufacturing ...)
- Truss TO:
  - problem usually parametrized via cross-section areas $\mathbf{a} \in \mathbb{R}^{n_\text{e}}_{\geq 0}$
  - $a_i > 0 \longleftrightarrow$ element i is present in the design
  - $a_i = 0 \longleftrightarrow$ element i is not present in the design
  - we do not know apriori whether $a_i > 0$ or $a_i = 0$

Ω $\tilde{\Omega}(\mathcal{N}, \mathcal{E})$ $\tilde{\Omega}(\mathcal{N}, \mathcal{E}^*)$



Compliance minimization: sizing

- Compliance: inverse measure of structural stiffness w.r.t. $\mathbf{f} \in \mathbb{R}^{n_{\mathrm{dof}}}$

$$c(\mathbf{a}) := \mathbf{f}^{\mathrm{T}}\mathbf{u}, \text{ where } \mathbf{K}(\mathbf{a})\mathbf{u} = \mathbf{f} \tag{1}$$

- Basic elastic-design formulation:

$$\min_{\mathbf{a}\in\mathbb{R}^{n_{\mathrm{e}}},\mathbf{u}\in\mathbb{R}^{n_{\mathrm{dof}}}} \quad \mathbf{f}^{\mathrm{T}}\mathbf{u} \tag{2a}$$

$$\text{subject to} \quad \mathbf{K}(\mathbf{a})\mathbf{u} = \mathbf{f}, \tag{2b}$$

$$\boldsymbol{\ell}^{\mathrm{T}}\mathbf{a} \leq \overline{V}, \tag{2c}$$

$$\mathbf{a} \geq \mathbf{0} \tag{2d}$$

- Why do we need $\boldsymbol{\ell}^{\mathrm{T}}\mathbf{a} \leq \overline{V}$?
- $\mathbf{a} \geq \mathbf{0} \implies \mathbf{K}(\mathbf{a}) \succeq 0$ ($\succeq$ means positive semi-definiteness)
- We assume $\forall \mathbf{a} > \mathbf{0} : \mathbf{K}(\mathbf{a}) \succ 0$ (positive definiteness)

- To avoid $\mathrm{Det}\left(\mathbf{K}(\mathbf{a})\right) = 0$, we set $\mathbf{a} \geq \varepsilon \mathbf{1} > \mathbf{0}$, where $\varepsilon \to 0$
- Then, $\mathbf{u} = \mathbf{K}(\mathbf{a})^{-1}\mathbf{f}$ is unique and can be eliminated from the formulation:

$$\min_{\mathbf{a}\in\mathbb{R}^{n_\mathrm{e}}} \quad \mathbf{f}^\mathrm{T}\mathbf{K}(\mathbf{a})^{-1}\mathbf{f} \tag{3a}$$

$$\text{subject to} \quad \boldsymbol{\ell}^\mathrm{T}\mathbf{a} \leq \overline{V}, \tag{3b}$$

$$\mathbf{a} \geq \varepsilon\mathbf{1} \tag{3c}$$

- **HW**: Show that the problem (3) is convex. Hint: Hessian matrix
- $\varepsilon \to 0$ impairs the conditioning of $\mathbf{K}(\mathbf{a})$
  - $\lambda_\mathrm{max}(\mathbf{K}(\mathbf{a}))$ driven by $\overline{V}$
  - $\lambda_\mathrm{min}(\mathbf{K}(\mathbf{a}))$ driven by ε

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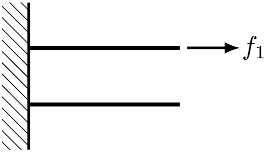

$$\mathbf{K} = \frac{E}{\ell}\begin{pmatrix} \frac{\overline{V} - \varepsilon\ell}{\ell} & 0 \\ 0 & \varepsilon \end{pmatrix}$$

$$\frac{\varepsilon}{\overline{V}} \to 0 \quad \longrightarrow \quad \mathrm{Cond}(\mathbf{K}) = \frac{\overline{V} - \varepsilon\ell}{\varepsilon\ell} \to \infty$$

Compliance minimization: potential-energy-based SDP

- Potential energy function (convex, minimum $-\mathbf{f}^{\mathrm{T}}\mathbf{u}$)

$$\Pi(\mathbf{a},\mathbf{u}) := \mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{a})\mathbf{u} - 2\mathbf{f}^{\mathrm{T}}\mathbf{u} \tag{4}$$

- Optimum design solves

$$\min_{\mathbf{a}\in\mathbb{R}^{n_\mathrm{e}}_{\geq 0}, \boldsymbol{\ell}^{\mathrm{T}}\mathbf{a}\leq\overline{V}} \min_{\mathbf{u}\in\mathbb{R}^{n_\mathrm{dof}}} \left(\mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{a})\mathbf{u} - 2\mathbf{f}^{\mathrm{T}}\mathbf{u}\right) \iff$$

$$\min_{\mathbf{a}\in\mathbb{R}^{n_\mathrm{e}}_{\geq 0}, \boldsymbol{\ell}^{\mathrm{T}}\mathbf{a}\leq\overline{V}} \max_{\mathbf{u}\in\mathbb{R}^{n_\mathrm{dof}}} \left(2\mathbf{f}^{\mathrm{T}}\mathbf{u} - \mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{a})\mathbf{u}\right)$$

- The inner max is equivalent to infinite number of inequalities

$$\begin{aligned}
\min_{\mathbf{a}\in\mathbb{R}^{n_\mathrm{e}}, c\in\mathbb{R}} \quad & c \\
\text{s.t.} \quad & c \geq \left(2\mathbf{f}^{\mathrm{T}}\mathbf{u} - \mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{a})\mathbf{u}\right), \forall \mathbf{u} \in \mathbb{R}^{n_\mathrm{dof}} \\
& \boldsymbol{\ell}^{\mathrm{T}}\mathbf{a} \leq \overline{V} \\
& \mathbf{a} \geq \mathbf{0}
\end{aligned}$$

- Potential energy function (convex, minimum $-\mathbf{f}^{\mathrm{T}}\mathbf{u}$)

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- Optimum design solves

$$\min_{\mathbf{a}\in\mathbb{R}^{n_\mathrm{e}}_{\geq 0}, \boldsymbol{\ell}^{\mathrm{T}}\mathbf{a}\leq\overline{V}} \ \min_{\mathbf{u}\in\mathbb{R}^{n_\mathrm{dof}}} \left(\mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{a})\mathbf{u} - 2\mathbf{f}^{\mathrm{T}}\mathbf{u}\right) \iff$$

$$\min_{\mathbf{a}\in\mathbb{R}^{n_\mathrm{e}}_{\geq 0}, \boldsymbol{\ell}^{\mathrm{T}}\mathbf{a}\leq\overline{V}} \ \max_{\mathbf{u}\in\mathbb{R}^{n_\mathrm{dof}}} \left(2\mathbf{f}^{\mathrm{T}}\mathbf{u} - \mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{a})\mathbf{u}\right)$$

- The inner max is equivalent to infinite number of inequalities

$$\begin{aligned}
\min_{\mathbf{a}\in\mathbb{R}^{n_\mathrm{e}}, c\in\mathbb{R}} \ & c \\
\text{s.t. } & c \geq \left(\frac{1}{\alpha}2\mathbf{f}^{\mathrm{T}}\mathbf{v} - \frac{1}{\alpha^2}\mathbf{v}^{\mathrm{T}}\mathbf{K}(\mathbf{a})\mathbf{v}\right), \forall \mathbf{v}\in\mathbb{R}^{n_\mathrm{dof}}, \forall\alpha\in\mathbb{R}\setminus\{0\} \\
& \boldsymbol{\ell}^{\mathrm{T}}\mathbf{a} \leq \overline{V} \\
& \mathbf{a} \geq \mathbf{0}
\end{aligned}$$

- Next, we rearrange the terms and multiply by α^2

$$\begin{aligned}\min_{\mathbf{a}\in\mathbb{R}^{n_\mathrm{e}},c\in\mathbb{R}} \quad & c \\ \text{s.t.} \quad & \alpha^2 c - 2\alpha\mathbf{f}^\mathrm{T}\mathbf{v} + \mathbf{v}^\mathrm{T}\mathbf{K}(\mathbf{a})\mathbf{v} \geq 0, \forall \mathbf{v}\in\mathbb{R}^{n_\mathrm{dof}}, \forall\alpha\in\mathbb{R}\setminus\{0\} \\ & \boldsymbol{\ell}^\mathrm{T}\mathbf{a} \leq \overline{V} \\ & \mathbf{a} \geq \mathbf{0}\end{aligned}$$

- We observe that the first inequality is actually

$$\begin{pmatrix}\alpha & \mathbf{v}^\mathrm{T}\end{pmatrix}\begin{pmatrix} c & -\mathbf{f}^\mathrm{T} \\ -\mathbf{f} & \mathbf{K}(\mathbf{a})\end{pmatrix}\begin{pmatrix}\alpha \\ \mathbf{v}\end{pmatrix} \geq 0, \underbrace{\forall \mathbf{v}\in\mathbb{R}^{n_\mathrm{dof}}, \forall\alpha\in\mathbb{R}\setminus\{0\}}_{\begin{pmatrix}\alpha \\ \mathbf{v}\end{pmatrix}\notin \mathrm{Ker}\begin{pmatrix} c & -\mathbf{f}^\mathrm{T} \\ -\mathbf{f} & \mathbf{K}(\mathbf{a})\end{pmatrix}}$$

- The inequality holds $\forall \mathbf{v}\in\mathbb{R}^{n_\mathrm{dof}}, \forall\alpha\in\mathbb{R} \implies$ we can write SDP formulation

- Linear SDP formulation:

$$\min_{\mathbf{a}\in\mathbb{R}^{n_\mathrm{e}},c\in\mathbb{R}} \quad c \tag{5a}$$

$$\text{subject to} \quad \begin{pmatrix} c & -\mathbf{f}^\mathrm{T} \\ -\mathbf{f} & \mathbf{K}(\mathbf{a}) \end{pmatrix} \succeq 0, \tag{5b}$$

$$\boldsymbol{\ell}^\mathrm{T}\mathbf{a} \leq \overline{V}, \tag{5c}$$

$$\mathbf{a} \geq \mathbf{0} \tag{5d}$$

- Convex problem
  - Let $\mathbf{X}$, $\mathbf{Y} \succeq 0$ and $\alpha \in [0,1]$. Then, $\alpha\mathbf{X} + (1-\alpha)\mathbf{Y} \succeq 0$:

$$\forall \mathbf{v}, \alpha : \quad \alpha\mathbf{v}^\mathrm{T}\mathbf{X}\mathbf{v} + (1-\alpha)\mathbf{v}^\mathrm{T}\mathbf{Y}\mathbf{v} \geq 0$$

- No specific assumptions on $\mathbf{K}(\mathbf{a})$ except for the linearity in $\mathbf{a}$
- Linear SDP solvers (MOSEK, SeDuMi, SDPA, ...)

Compliance minimization: potential-energy-based QP

- Potential energy function (convex, minimum $-\mathbf{f}^{\mathrm{T}}\mathbf{u}$)

$$\Pi(\mathbf{a},\mathbf{u}) := \mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{a})\mathbf{u} - 2\mathbf{f}^{\mathrm{T}}\mathbf{u} \tag{6}$$

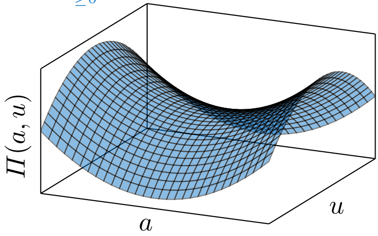
- Optimum design solves

$$\min_{\mathbf{a}\in\mathbb{R}^{n_\mathrm{e}}_{\geq 0},\boldsymbol{\ell}^{\mathrm{T}}\mathbf{a}\leq\overline{V}} \min_{\mathbf{u}\in\mathbb{R}^{n_\mathrm{dof}}} \left(\mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{a})\mathbf{u} - 2\mathbf{f}^{\mathrm{T}}\mathbf{u}\right) \iff$$

$$\min_{\mathbf{a}\in\mathbb{R}^{n_\mathrm{e}}_{\geq 0},\boldsymbol{\ell}^{\mathrm{T}}\mathbf{a}\leq\overline{V}} \max_{\mathbf{u}\in\mathbb{R}^{n_\mathrm{dof}}} \left(2\mathbf{f}^{\mathrm{T}}\mathbf{u} - \mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{a})\mathbf{u}\right)$$

- By saddle point theorem, order min and max can be switched

$$\max_{\mathbf{u}\in\mathbb{R}^{n_\mathrm{dof}}} \min_{\mathbf{a}\in\mathbb{R}^{n_\mathrm{e}}_{\geq 0},\boldsymbol{\ell}^{\mathrm{T}}\mathbf{a}\leq\overline{V}} \left(2\mathbf{f}^{\mathrm{T}}\mathbf{u} - \mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{a})\mathbf{u}\right)$$



- $\min_{\mathbf{a}\in\mathbb{R}^{n_\mathrm{e}}_{\geq 0}, \boldsymbol{\ell}^\mathrm{T}\mathbf{a}\leq\overline{V}} \left(2\mathbf{f}^\mathrm{T}\mathbf{u} - \sum_{i=1}^{n_\mathrm{e}} \left[\mathbf{u}^\mathrm{T}\mathbf{K}_i(a_i)\mathbf{u}\right]\right)$ is a linear program
- Optimal solution: assign all $\overline{V}$ to elements with maximum specific strain energy $\mathbf{u}^\mathrm{T}\mathbf{b}_i^\mathrm{T}\frac{E_i}{\ell_i^2}\mathbf{b}_i\mathbf{u}$ (remark: $\mathbf{K}_i(a_i)$ not required to be rank 1)

$$\max_{\mathbf{u}\in\mathbb{R}^{n_\mathrm{dof}}} \left(2\mathbf{f}^\mathrm{T}\mathbf{u} - \overline{V}\max_i \left\{\mathbf{u}^\mathrm{T}\mathbf{b}_i^\mathrm{T}\frac{E_i}{\ell_i^2}\mathbf{b}_i\mathbf{u}\right\}\right)$$

- We introduce a slack variable $\beta\in\mathbb{R}$ to avoid the non-differentiable max operator

$$\max_{\mathbf{u}\in\mathbb{R}^{n_\mathrm{dof}}, \beta\in\mathbb{R}} \left(2\mathbf{f}^\mathrm{T}\mathbf{u} - \overline{V}\beta\right) \tag{7a}$$

$$\text{subject to } \mathbf{u}^\mathrm{T}\mathbf{b}_i^\mathrm{T}\frac{E_i}{\ell_i^2}\mathbf{b}_i\mathbf{u} \leq \beta, \quad \forall i\in\{1,\dots,n_\mathrm{e}\} \tag{7b}$$

- Lagrangian function

$$\mathcal{L}(\mathbf{u},\beta,\boldsymbol{\lambda}) = -2\mathbf{f}^{\mathrm{T}}\mathbf{u} + \overline{V}\beta + \sum_{i=1}^{n_{\mathrm{e}}}\left[\lambda_i\left(\mathbf{u}^{\mathrm{T}}\mathbf{b}_i^{\mathrm{T}}\frac{E_i}{\ell_i^2}\mathbf{b}_i\mathbf{u} - \beta\right)\right]$$

- Karush-Kuhn-Tucker conditions (Slater's condition satisfied)

stationarity:
$$\frac{\partial\mathcal{L}}{\partial\mathbf{u}} = -2\mathbf{f} + 2\left(\sum_{i=1}^{n_{\mathrm{e}}}\lambda_i^*\mathbf{b}_i^{\mathrm{T}}\frac{E_i}{\ell_i^2}\mathbf{b}_i\right)\mathbf{u}^* = \mathbf{0}$$

$$\frac{\partial\mathcal{L}}{\partial\beta} = \overline{V} - \sum_{i=1}^{n_{\mathrm{e}}}\lambda_i^* = 0$$

primal feasibility:
$$0 \geq (\mathbf{u}^*)^{\mathrm{T}}\mathbf{b}_i^{\mathrm{T}}\frac{E_i}{\ell_i^2}\mathbf{b}_i\mathbf{u}^* - \beta^*$$

dual feasibility:
$$\mathbf{0} \leq \boldsymbol{\lambda}^*$$

complementary slackness:
$$\mathbf{0} = \sum_{i=1}^{n_{\mathrm{e}}}\lambda_i^*\left((\mathbf{u}^*)^{\mathrm{T}}\mathbf{b}_i^{\mathrm{T}}\frac{E_i}{\ell_i^2}\mathbf{b}_i\mathbf{u}^* - \beta^*\right)$$

- What is the physical interpretation of λ_i^*?

- Lagrangian function

$$\mathcal{L}(\mathbf{u}, \beta, \boldsymbol{\lambda}) = -2\mathbf{f}^{\mathrm{T}}\mathbf{u} + \overline{V}\beta + \sum_{i=1}^{n_{\mathrm{e}}} \left[\lambda_i \left(\mathbf{u}^{\mathrm{T}}\mathbf{b}_i^{\mathrm{T}}\frac{E_i}{\ell_i^2}\mathbf{b}_i\mathbf{u} - \beta\right)\right]$$

- Karush-Kuhn-Tucker conditions (Slater's condition satisfied)

stationarity: $$\frac{\partial \mathcal{L}}{\partial \mathbf{u}} = -2\mathbf{f} + 2\left(\sum_{i=1}^{n_{\mathrm{e}}} \lambda_i^* \mathbf{b}_i^{\mathrm{T}}\frac{E_i}{\ell_i^2}\mathbf{b}_i\right)\mathbf{u}^* = \mathbf{0}$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = \overline{V} - \sum_{i=1}^{n_{\mathrm{e}}} \lambda_i^* = 0$$

primal feasibility: $$0 \geq (\mathbf{u}^*)^{\mathrm{T}}\mathbf{b}_i^{\mathrm{T}}\frac{E_i}{\ell_i^2}\mathbf{b}_i\mathbf{u}^* - \beta^*$$

dual feasibility: $$\mathbf{0} \leq \boldsymbol{\lambda}^*$$

complementary slackness: $$\mathbf{0} = \sum_{i=1}^{n_{\mathrm{e}}} \lambda_i^* \left((\mathbf{u}^*)^{\mathrm{T}}\mathbf{b}_i^{\mathrm{T}}\frac{E_i}{\ell_i^2}\mathbf{b}_i\mathbf{u}^* - \beta^*\right)$$

- At the optimum, we have $\lambda_i^* = \ell_i a_i^*$

Compliance minimization: potential-energy-based LP

- Exploiting that for single load case and no additional constraints on $\mathbf{a}$ (lower bounds/upper bounds/mapping), $\mathbf{K}_i(a_i)$ is of rank 1
- Put $\gamma^2 = \beta$ to (7)[1]

$$\max_{\mathbf{u}\in\mathbb{R}^{n_{\mathrm{dof}}},\gamma\in\mathbb{R}} \left(2\mathbf{f}^{\mathrm{T}}\mathbf{u} - \overline{V}\gamma^2\right) \tag{8a}$$

$$\text{subject to } \mathbf{u}^{\mathrm{T}}\mathbf{b}_i^{\mathrm{T}}\frac{E_i}{\ell_i^2}\mathbf{b}_i\mathbf{u} \leq \gamma^2, \quad \forall i \in \{1,\dots,n_{\mathrm{e}}\} \tag{8b}$$

- Constraint (8b) can be rewritten as

$$\left(\frac{\sqrt{E_i}}{\ell_i}\mathbf{b}_i\mathbf{u}\right)^2 \leq \gamma^2, \quad \forall i \in \{1,\dots,n_{\mathrm{e}}\} \iff$$

$$-\gamma \leq \frac{\sqrt{E_i}}{\ell_i}\mathbf{b}_i\mathbf{u} \leq \gamma, \quad \forall i \in \{1,\dots,n_{\mathrm{e}}\}$$

- Fix $\gamma = 1 \rightarrow$ we expect the dual multipliers to be ratios of bar volumes

[1] Then, $\lambda_i^* = 2\gamma^*\ell_i a_i^*$

- We arrive at a linear program

$$\max_{\mathbf{u}\in\mathbb{R}^{n_{\mathrm{dof}}}} \quad \mathbf{f}^{\mathrm{T}}\mathbf{u} \tag{9a}$$

$$\text{subject to} \quad \frac{\sqrt{E_i}}{\ell_i}\mathbf{b}_i\mathbf{u} \leq 1, \quad \forall i \in \{1,\ldots,n_{\mathrm{e}}\} \tag{9b}$$

$$\frac{\sqrt{E_i}}{\ell_i}\mathbf{b}_i\mathbf{u} \geq -1, \quad \forall i \in \{1,\ldots,n_{\mathrm{e}}\} \tag{9c}$$

- Lagrangian function

$$\mathcal{L}(\mathbf{u},\boldsymbol{\lambda}_+,\boldsymbol{\lambda}_-) = -\mathbf{f}^{\mathrm{T}}\mathbf{u} + \sum_{i=1}^{n_{\mathrm{e}}}\lambda_{+,i}\left(\frac{\sqrt{E_i}}{\ell_i}\mathbf{b}_i\mathbf{u} - 1\right) + \sum_{i=1}^{n_{\mathrm{e}}}\lambda_{-,i}\left(-\frac{\sqrt{E_i}}{\ell_i}\mathbf{b}_i\mathbf{u} - 1\right)$$

- KKT conditions:

$$\text{stationarity:}\quad \frac{\partial \mathcal{L}}{\partial \mathbf{u}} = -\mathbf{f} + \sum_{i=1}^{n_\mathrm{e}} \left(\lambda^*_{+,i} - \lambda^*_{-,i}\right) \frac{\sqrt{E_i}}{\ell_i} \mathbf{b}_i^\mathrm{T} = \mathbf{0}$$

$$\text{primal feas.:}\quad -1 \leq \frac{\sqrt{E_i}}{\ell_i} \mathbf{b}_i \hat{\mathbf{u}}^* \leq 1, \quad \forall i \in \{1, \dots, n_\mathrm{e}\}$$

$$\text{dual feas.:}\quad \boldsymbol{\lambda}^*_+ \geq \mathbf{0}$$

$$\boldsymbol{\lambda}^*_- \geq \mathbf{0}$$

$$\text{compl. slack.:}\quad 0 = \sum_{i=1}^{n_\mathrm{e}} \left[\lambda^*_{+,i} \left(\frac{\sqrt{E_i}}{\ell_i} \mathbf{b}_i \hat{\mathbf{u}}^* - 1\right) - \lambda^*_{-,i} \left(\frac{\sqrt{E_i}}{\ell_i} \mathbf{b}_i \hat{\mathbf{u}}^* + 1\right)\right]$$

- Stationarity implies static equilibrium $\mathbf{B}^\mathrm{T} \left(\mathbf{s}_+^\mathrm{T} \quad -\mathbf{s}_-^\mathrm{T}\right)^\mathrm{T} = \mathbf{f}$, where $s_{+,i} = \frac{\sqrt{E_i}\lambda^*_{+,i}}{\ell_i}$ and $s_{-,i} = \frac{\sqrt{E_i}\lambda^*_{-,i}}{\ell_i}$
- $\sqrt{E_i}$ can be interpreted as a (dimensionless) yield stress $\sigma_{0,i}$ and $(\lambda_{i,+} + \lambda_{i,-})$ as (scaled) bar volume
- Unscaling: $a_i^* = \frac{\lambda^*_{+,i} + \lambda^*_{-,i}}{\ell_i} \frac{\overline{V}}{\mathbf{1}^\mathrm{T}\boldsymbol{\lambda}^*_+ + \mathbf{1}^\mathrm{T}\boldsymbol{\lambda}^*_-}$, $\mathbf{u}^* = \hat{\mathbf{u}}^* \frac{\mathbf{1}^\mathrm{T}\boldsymbol{\lambda}^*_+ + \mathbf{1}^\mathrm{T}\boldsymbol{\lambda}^*_-}{\overline{V}}$

Compliance minimization: complementary-strain-energy-based LP

- Lagrangian function

$$\mathcal{L}(\mathbf{u}, \boldsymbol{\lambda}_+, \boldsymbol{\lambda}_-) = -\mathbf{f}^{\mathrm{T}}\mathbf{u} + \sum_{i=1}^{n_{\mathrm{e}}} \lambda_{+,i} \left(\frac{\sqrt{E_i}}{\ell_i} \mathbf{b}_i \mathbf{u} - 1 \right) + \sum_{i=1}^{n_{\mathrm{e}}} \lambda_{-,i} \left(-\frac{\sqrt{E_i}}{\ell_i} \mathbf{b}_i \mathbf{u} - 1 \right)$$

- Lagrange dual function

$$d(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_-) = \inf_{\mathbf{u} \in \mathbb{R}^{n_{\mathrm{dof}}}} \left[\left(-\mathbf{f}^{\mathrm{T}} + \sum_{i=1}^{n_{\mathrm{e}}} \lambda_{+,i} \frac{\sqrt{E_i}}{\ell_i} \mathbf{b}_i - \sum_{i=1}^{n_{\mathrm{e}}} \lambda_{-,i} \frac{\sqrt{E_i}}{\ell_i} \mathbf{b}_i \right) \mathbf{u} - \sum_{i=1}^{n_{\mathrm{e}}} \lambda_{+,i} - \sum_{i=1}^{n_{\mathrm{e}}} \lambda_{-,i} \right]$$

- Lagrangian function

$$\mathcal{L}(\mathbf{u}, \boldsymbol{\lambda}_+, \boldsymbol{\lambda}_-) = -\mathbf{f}^{\mathrm{T}}\mathbf{u} + \sum_{i=1}^{n_{\mathrm{e}}} \lambda_{+,i} \left(\frac{\sqrt{E_i}}{\ell_i} \mathbf{b}_i \mathbf{u} - 1 \right) + \sum_{i=1}^{n_{\mathrm{e}}} \lambda_{-,i} \left(-\frac{\sqrt{E_i}}{\ell_i} \mathbf{b}_i \mathbf{u} - 1 \right)$$

- Lagrange dual function after substituting $s_{+,i} = \frac{\sqrt{E_i}\lambda^*_{+,i}}{\ell_i}$ and $s_{-,i} = \frac{\sqrt{E_i}\lambda^*_{-,i}}{\ell_i}$

$$d(\mathbf{s}_+, \mathbf{s}_-) = \inf_{\mathbf{u} \in \mathbb{R}^{n_{\mathrm{dof}}}} \left[\left(-\mathbf{f}^{\mathrm{T}} + \sum_{i=1}^{n_{\mathrm{e}}} \left[s_{+,i} - s_{-,i} \right] \mathbf{b}_i \right) \mathbf{u} - \sum_{i=1}^{n_{\mathrm{e}}} \frac{\ell_i (\mathbf{s}_+ + \mathbf{s}_-)}{\sqrt{E_i}} \right]$$

- Lagrangian function

$$\mathcal{L}(\mathbf{u}, \boldsymbol{\lambda}_+, \boldsymbol{\lambda}_-) = -\mathbf{f}^{\mathrm{T}}\mathbf{u} + \sum_{i=1}^{n_\mathrm{e}} \lambda_{+,i} \left(\frac{\sqrt{E_i}}{\ell_i} \mathbf{b}_i \mathbf{u} - 1 \right) + \sum_{i=1}^{n_\mathrm{e}} \lambda_{-,i} \left(-\frac{\sqrt{E_i}}{\ell_i} \mathbf{b}_i \mathbf{u} - 1 \right)$$

- Lagrange dual function after substituting $s_{+,i} = \frac{\sqrt{E_i}\lambda^*_{+,i}}{\ell_i}$ and $s_{-,i} = \frac{\sqrt{E_i}\lambda^*_{-,i}}{\ell_i}$ and $\sqrt{E}_i = \sigma_{0,i}$

$$d(\mathbf{s}_+, \mathbf{s}_-) = \inf_{\mathbf{u} \in \mathbb{R}^{n_\mathrm{dof}}} \left[\left(-\mathbf{f}^{\mathrm{T}} + \sum_{i=1}^{n_\mathrm{e}} [s_{+,i} - s_{-,i}] \, \mathbf{b}_i \right) \mathbf{u} - \sum_{i=1}^{n_\mathrm{e}} \frac{\ell_i (s_{+,i} + s_{-,i})}{\sigma_{0,i}} \right]$$

- Lagrangian dual formulation: we search for $\sup_{\mathbf{s}_+ \in \mathbb{R}^{n_\mathrm{e}}, \mathbf{s}_- \in \mathbb{R}^{n_\mathrm{e}}} d(\mathbf{s}_+, \mathbf{s}_-)$

$$\min_{\mathbf{s}_+ \in \mathbb{R}^{n_\mathrm{e}}, \mathbf{s}_- \in \mathbb{R}^{n_\mathrm{e}}} \quad \sum_{i=1}^{n_\mathrm{e}} \frac{s_{+,i} + s_{-,i}}{\sigma_{0,i}} \ell_i \tag{10a}$$

$$\text{subject to} \quad \mathbf{B}^\mathrm{T} \begin{pmatrix} \mathbf{s}_+^\mathrm{T} & -\mathbf{s}_-^\mathrm{T} \end{pmatrix}^\mathrm{T} = \mathbf{f} \tag{10b}$$

$$\mathbf{s}_+ \geq \mathbf{0} \tag{10c}$$

$$\mathbf{s}_- \geq \mathbf{0} \tag{10d}$$

- Objective function equivalent to $\boldsymbol{\ell}^\mathrm{T} \mathbf{a}$ (substitution possible)
- Optimum design is fully stressed (by assumptions) and statically determinate (unique solution to static equation, compatibility satisfied automatically)

Compliance minimization: complementary-strain-energy-based SOCP

- Complementary strain energy function (1/2 removed to stay consistent)

$$\Pi_2(\mathbf{a},\mathbf{s}) = \sum_{i=1}^{n_\mathrm{e}} \int_{v_\mathrm{e}} E_i \varepsilon_i(a_i, s_i)^2 \mathrm{d}v = \sum_{i=1}^{n_\mathrm{e}} \frac{\ell_i s_i^2}{E_i a_i} \quad \text{such that } \mathbf{B}^\mathrm{T}\mathbf{s} = \mathbf{f}$$

- Optimum design solves

$$\inf_{\mathbf{a}\in\mathbb{R}^{n_\mathrm{e}}_{\geq 0}, \boldsymbol{\ell}^\mathrm{T}\mathbf{a}\leq\overline{V}} \min_{\mathbf{s}\in\mathbb{R}^{n_\mathrm{e}}} \sum_{i=1}^{n_\mathrm{e}} \frac{\ell_i s_i^2}{E_i a_i} \tag{11a}$$

$$\text{subject to} \quad \mathbf{B}^\mathrm{T}\mathbf{s} = \mathbf{f} \tag{11b}$$

- We need inf because of $a_i = 0$
- Define $w_i \in \mathbb{R}$ to be the energy of the i-th bar

$$\inf_{\mathbf{a}\in\mathbb{R}^{n_\mathrm{e}}_{\geq 0}, \boldsymbol{\ell}^\mathrm{T}\mathbf{a}\leq\overline{V}} \min_{\mathbf{s}\in\mathbb{R}^{n_\mathrm{e}}, \mathbf{w}\in\mathbb{R}^{n_\mathrm{e}}} \sum_{i=1}^{n_\mathrm{e}} w_i$$

$$\text{subject to} \quad \mathbf{B}^\mathrm{T}\mathbf{s} = \mathbf{f}$$

$$w_i \geq \frac{\ell_i s_i^2}{E_i a_i}, \quad \forall i \in \{1, \ldots, n_\mathrm{e}\}$$

- To eliminate inf, we must take care of $a_i = 0$

$$4w_i a_i \geq 4\frac{\ell_i s_i^2}{E_i}$$

- Add $w_i^2 - 2w_i a_i + a_i^2$ to both sides of the inequality

$$(w_i + a_i)^2 \geq 4\frac{\ell_i s_i^2}{E_i} + (w_i - a_i)^2$$

- This is equivalent to $w_i = w_i^- + w_i^+$ and $a_i = a_i^- + a_i^+$

$$w_i^+ + a_i^+ \geq \left\| \begin{pmatrix} 2\sqrt{\frac{\ell_i}{E_i}} s_i & w_i^+ - a_i^+ \end{pmatrix} \right\|_2$$
$$-w_i^- - a_i^- \geq \left\| \begin{pmatrix} 2\sqrt{\frac{\ell_i}{E_i}} s_i & w_i^- - a_i^- \end{pmatrix} \right\|_2,$$

- The second constraint is redundant because $\mathbf{a} \geq \mathbf{0}$ and $\mathbf{w} \geq \mathbf{0}$ by definition

- Final SOCP formulation:

$$\min_{\mathbf{a}\in\mathbb{R}^{n_\mathrm{e}},\mathbf{s}\in\mathbb{R}^{n_\mathrm{e}},\mathbf{w}\in\mathbb{R}^{n_\mathrm{e}}} \quad \mathbf{1}^\mathrm{T}\mathbf{w} \tag{12a}$$

$$\text{subject to } \mathbf{B}^\mathrm{T}\mathbf{s} = \mathbf{f} \tag{12b}$$

$$w_i + a_i \geq \left\| \begin{pmatrix} 2\sqrt{\frac{\ell_i}{E_i}} s_i & w_i - a_i \end{pmatrix} \right\|_2, \quad \forall i \in \{1,\dots,n_\mathrm{e}\} \tag{12c}$$

$$\boldsymbol{\ell}^\mathrm{T}\mathbf{a} \leq \overline{V} \tag{12d}$$

$$\mathbf{a} \geq \mathbf{0} \tag{12e}$$

- What is the relation to the minimum-volume LP?
- What is the relation to the SDP formulation?
- Convex: Let $t, u \geq 0$, $\|\mathbf{x}\|_2 \leq t$, $\|\mathbf{y}\|_2 \leq u$

$$\alpha\|\mathbf{x}\|_2 + (1-\alpha)\|\mathbf{y}\|_2 \leq \alpha t + (1-\alpha)u$$

$$\|\alpha\mathbf{x}\|_2 + \|(1-\alpha)\mathbf{y}\|_2 \leq \alpha t + (1-\alpha)u$$

by Cauchy-Schwarz ineq.: $$\|\alpha\mathbf{x} + (1-\alpha)\mathbf{y}\|_2 \leq \alpha t + (1-\alpha)u$$

Extensions

- Free-vibrations ($\lambda = 4\pi^2 f^2$):

$$\mathbf{K}(\mathbf{a})\mathbf{u} - \lambda\mathbf{M}(\mathbf{a})\mathbf{u} = \mathbf{0}$$

- Rayleigh principle

$$\lambda_{\min} = \inf_{\mathbf{u}\in\mathbb{R}^{n_{\text{dof}}}\setminus\text{Ker}(\mathbf{M}(\mathbf{a}))} \frac{\mathbf{u}^{\text{T}}\mathbf{K}(\mathbf{a})\mathbf{u}}{\mathbf{u}^{\text{T}}\mathbf{M}(\mathbf{a})\mathbf{u}}$$

$$\lambda_{\min} \leq \frac{\mathbf{u}^{\text{T}}\mathbf{K}(\mathbf{a})\mathbf{u}}{\mathbf{u}^{\text{T}}\mathbf{M}(\mathbf{a})\mathbf{u}}, \quad \forall\mathbf{u} \in \mathbb{R}^{n_{\text{dof}}} \setminus \text{Ker}(\mathbf{M}(\mathbf{a}))$$

$$0 \leq \mathbf{u}^{\text{T}}\mathbf{K}(\mathbf{a})\mathbf{u} - \lambda_{\min}\mathbf{u}^{\text{T}}\mathbf{M}(\mathbf{a})\mathbf{u}, \quad \forall\mathbf{u} \in \mathbb{R}^{n_{\text{dof}}} \setminus \text{Ker}(\mathbf{M}(\mathbf{a}))$$

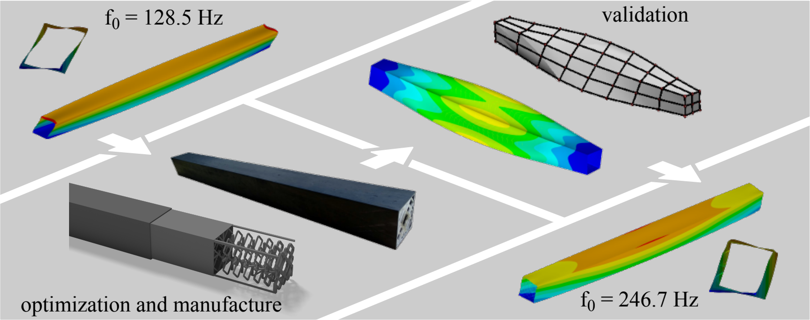
- Set $\mathbf{u} \in \text{Ker}(\mathbf{M}(\mathbf{a}))$: the inequality remains valid

$$\mathbf{u}^{\text{T}}\mathbf{K}(\mathbf{a})\mathbf{u} - \lambda_{\min}\mathbf{u}^{\text{T}}\mathbf{M}(\mathbf{a})\mathbf{u} \geq 0, \quad \forall\mathbf{u} \in \mathbb{R}^{n_{\text{dof}}}$$

$$\mathbf{K}(\mathbf{a}) - \lambda_{\min}\mathbf{M}(\mathbf{a}) \succeq 0$$

- Eigenvalue maximization: quasi-convex problem solvable by bisection

- Industrial application: decreasing elastic wall instabilities of thin-walled carbon fiber beams by designing minimum-weight truss stiffeners



- Instead of optimizing over fixed $\mathbf{f} \in \mathbb{R}^{n_{\mathrm{dof}}}$, consider optimizing over the worst-case load from the ellipsoid $\mathbf{f} = \mathbf{Q}\mathbf{e}$ with $\mathbf{e}^{\mathrm{T}}\mathbf{e} \leq 1$ and $\mathbf{Q} \in \mathbb{R}^{n_{\mathrm{dof}} \times q}$ and $\mathbf{e} \in \mathbb{R}^{q}$
- Potential energy function

$$\min_{\mathbf{a}\in\mathbb{R}^{n_{\mathrm{e}}}_{\geq 0}, \boldsymbol{\ell}^{\mathrm{T}}\mathbf{a} \leq \overline{V}} \; \max_{\mathbf{e}^{\mathrm{T}}\mathbf{e}\leq 1} \; \max_{\mathbf{u}\in\mathbb{R}^{n_{\mathrm{dof}}}} \; 2\left(\mathbf{Q}\mathbf{e}\right)^{\mathrm{T}}\mathbf{u} - \mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{a})\mathbf{u}$$

- The inner $\max$ operators are equivalent to infinite number of matrix inequalities

$$\begin{aligned}
\min_{\mathbf{a}\in\mathbb{R}^{n_{\mathrm{e}}}, c\in\mathbb{R}} \; & c \\
\text{s.t. } & c \geq \left(2\left(\mathbf{Q}\mathbf{e}\right)^{\mathrm{T}}\mathbf{u} - \mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{a})\mathbf{u}\right), \forall \mathbf{u} \in \mathbb{R}^{n_{\mathrm{dof}}}, \forall \mathbf{e} \in \mathbb{R}^{q} : \mathbf{e}^{\mathrm{T}}\mathbf{e} \leq 1 \\
& \boldsymbol{\ell}^{\mathrm{T}}\mathbf{a} \leq \overline{V} \\
& \mathbf{a} \geq \mathbf{0}
\end{aligned}$$

- Thus, we also have for $\gamma > 0$

  $\gamma^2 c - 2\left(\mathbf{Q}\gamma\mathbf{e}\right)^{\mathrm{T}} \gamma\mathbf{u} + \gamma\mathbf{u}^{\mathrm{T}}\mathbf{K}(\mathbf{a})\gamma\mathbf{u} \geq 0, \forall\mathbf{u} \in \mathbb{R}^{n_{\mathrm{dof}}}, \forall\mathbf{e} \in \mathbb{R}^{q} : \mathbf{e}^{\mathrm{T}}\mathbf{e} \leq 1$
- Set $\gamma\mathbf{e} = \mathbf{t}$ and $\gamma\mathbf{u} = \mathbf{w}$

  $\gamma^2 c - 2\left(\mathbf{Q}\mathbf{t}\right)^{\mathrm{T}} \mathbf{w} + \mathbf{w}^{\mathrm{T}}\mathbf{K}(\mathbf{a})\mathbf{w} \geq 0, \forall\mathbf{w} \in \mathbb{R}^{n_{\mathrm{dof}}}, \forall\mathbf{t} \in \mathbb{R}^{q} : \mathbf{t}^{\mathrm{T}}\mathbf{t} \leq \gamma^2$
- Finally, we obtain

$$\mathbf{t}^{\mathrm{T}}\mathbf{t}c - 2\left(\mathbf{Q}\mathbf{t}\right)^{\mathrm{T}} \mathbf{w} + \mathbf{w}^{\mathrm{T}}\mathbf{K}(\mathbf{a})\mathbf{w} \geq 0 \iff \begin{pmatrix} c\mathbf{I} & -\mathbf{Q}^{\mathrm{T}} \\ -\mathbf{Q} & \mathbf{K}(\mathbf{a}) \end{pmatrix} \succeq 0$$

and thus the linear SDP

$$\begin{aligned} \min_{\mathbf{a}\in\mathbb{R}^{n_{\mathrm{e}}}, c\in\mathbb{R}} \quad & c \\ \text{s.t.} \quad & \begin{pmatrix} c\mathbf{I} & -\mathbf{Q}^{\mathrm{T}} \\ -\mathbf{Q} & \mathbf{K}(\mathbf{a}) \end{pmatrix} \succeq 0 \\ & \boldsymbol{\ell}^{\mathrm{T}}\mathbf{a} \leq \overline{V} \\ & \mathbf{a} \geq \mathbf{0} \end{aligned}$$

- Other extensions:
  - Multiple loading conditions
    - Can be incorporated in QP, SOCP, SDP formulations
    - Not in LP!
  - Power of in-phase harmonic vibrations (linear SDP)
  - Linear buckling (nonlinear SDP)
  - Robust optimization (linear SDP)
  - Displacement constraints (nonconvex)
  - Stress constraints (nonconvex)
  - Manufacturability constraints

Summary

Summary

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- High stiffness/weight ratio
- Many convex formulations
- Fast solution
- Straightforward implementation

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- Not everything behaves as trusses
- Some constraints ruin the problem structure
- Collinear members
- Singularity in straight bars with refined discretization

Literature

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