Cross-section kinematics assumptions

- Distributed load acts in the $xz$ plane, which is also a plane of symmetry of a body $\Omega \Rightarrow v(x) = 0\ m$

- Vertical displacement does not vary along the height of the beam (when compared to the value of the displacement) $\Rightarrow w(x) = w(x)$.

- The cross sections remain planar but \textit{not necessarily perpendicular} to the deformed beam axis $\Rightarrow u(x) = u(x, z) = \varphi_y(x)z$
These hypotheses were independently proposed by Timoshenko [6], Reissner [5] and Mindlin [4].

2 Strain-displacement equations

Cross-section kinematics assumptions imply that only non-zero strain components are

\[
\varepsilon_x(x) = \frac{\partial u(x)}{\partial x} = \frac{\partial}{\partial x} (\varphi_y(x)z) = \frac{d \varphi_y(x)}{dx} z = \kappa_y(x)z
\]

\[
\gamma_{zx}(x) = \frac{\partial w(x)}{\partial x} + \frac{\partial u(x)}{\partial z} = \frac{dw(x)}{dx} + \frac{\partial}{\partial z} (\varphi_y(x)z) = \frac{dw(x)}{dx} + \varphi_y(x),
\]
when \( \kappa_y \) denotes the pseudo-curvature of the deformed beam centerline.

<table>
<thead>
<tr>
<th>Bernoulli-Navier [7, kap. II.2]</th>
<th>Mindlin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Valid for</td>
<td></td>
</tr>
<tr>
<td>( h/L &lt; 1/10 )</td>
<td>( h/L &lt; 1/3 )</td>
</tr>
<tr>
<td>Cross-section</td>
<td>planar, perpendicular</td>
</tr>
<tr>
<td>( \gamma_{zx} )</td>
<td>0</td>
</tr>
<tr>
<td>Unknowns</td>
<td>( w(x) )</td>
</tr>
<tr>
<td></td>
<td>( \varphi_y(x) = -\frac{dw(x)}{dx} )</td>
</tr>
</tbody>
</table>

### 3 Stress-strain relations

- For simplicity, we will assume \( \varepsilon_0 = 0 \)

\[
\sigma_x(x, z) = E(x)\varepsilon_x(x, z) = E(x)\kappa_y(x)z
\]

\[
\tau_{zx}(x) = G(x)\gamma_{zx}(x) = G(x)\left(\frac{dw(x)}{dx} + \varphi_y(x)\right)
\]
• Non-zero internal forces:

\[ M_y(x) = \int_{A(x)} \sigma_x(x, z) \, dy \, dz = E(x) \kappa_y(x) \int_{A(x)} z^2 \, dy \, dz \]

\[ = E(x) I_y(x) \kappa_y(x) = E(x) I_y(x) \frac{d\varphi_y(x)}{dx} \]  

(1)

\[ Q_z^c(x) = \int_{A(x)} \tau_{zx}(x) \, dy \, dz = G(x) \left( \frac{dw(x)}{dx} + \varphi_y(x) \right) \int_{A(x)} \, dy \, dz \]

\[ = G(x) A(x) \left( \frac{dw(x)}{dx} + \varphi_y(x) \right) \]

• Distribution of shear stresses \( \tau_{zx} \) for a rectangular cross-section

<table>
<thead>
<tr>
<th></th>
<th>Bernoulli-Navier</th>
<th>Mindlin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constitutive eqs: ( \tau = G\gamma )</td>
<td>0</td>
<td>constant</td>
</tr>
<tr>
<td>Equilibrium eqs</td>
<td>quadratic</td>
<td>?</td>
</tr>
</tbody>
</table>

[7, kap. II.2.5]

• Therefore, we modify the shear force relation in order to take into
account equilibrium equations, at least in the sense of average work of shear components

\[ Q_z(x) = k(x)Q_z^c(x) = k(x)G(x)A(x) \left( \frac{dw(x)}{dx} + \varphi_y(x) \right) \]  

(2)

- The multiplier \( k(x) \) depends on a shape of a cross-section, for a rectangular cross-section, \( k = 5/6 \).

**Homework 1.** Derive the relation for the constant \( k \) for a general cross-section: \( k = I^2_y / (A \int_A \frac{S^2_y(z)}{b^2(z)} \, dA) \).
4 Equilibrium equations

- Equilibrium equation of vertical forces (a)
\[
\frac{dQ_z(x)}{dx} + \bar{f}_z(x) = 0
\]  
(3)

- Equilibrium equation of moments (b)
\[
\frac{dM_y(x)}{dx} - Q_z(x) = 0
\]  
(4)

- For a detailed derivation see Lecture 1, Homework 1.
5 Governing equations

\[
\frac{d}{dx} \left( k(x)G(x)A(x) \left( \frac{dw(x)}{dx} + \varphi_y(x) \right) \right) + f_z(x) = 0 \quad (5)
\]

\[
\frac{d}{dx} \left( E(x)I_y(x) \frac{d\varphi_y(x)}{dx} \right) - k(x)G(x)A(x) \left( \frac{dw(x)}{dx} + \varphi_y(x) \right) = 0 \quad (6)
\]

5.1 Kinematic boundary conditions: \( x \in I_u \)

Pinned end: \( w = 0 \)

Clamped end: \( w = 0, \varphi_y = 0 \)

5.2 Static boundary conditions: \( x \in I_p \)

\[ Q_z(x) = \overline{Q_z(x)} \quad , \quad M_y(x) = \overline{M_y(x)} \]
6 Weak solution

- For notational simplicity, we will use relations (3)–(4) instead of (5)–(6).

- We will “weight” Eq. (3) by term $\delta w(x)$, Eq. (4) by $\delta \varphi_y(x)$ and integrate them on $I$. This leads to conditions

$$0 = \int_I \delta w(x) \left( \frac{dQ_z(x)}{dx} + f_z(x) \right) \, dx,$$

$$0 = \int_I \delta \varphi_y(x) \left( \frac{dM_y(x)}{dx} - Q_z(x) \right) \, dx,$$

which are to be satisfied for all $\delta w(x)$ and $\delta \varphi_y(x)$ compatible with the kinematic boundary conditions.
• By parts integration

\[
0 = [\delta w(x)Q_z(x)]_a^b - \int_I \frac{d(\delta w(x))}{dx} Q_z(x) \, dx + \int_I \delta w(x) f_z(x) \, dx
\]

\[
0 = [\delta \varphi_y(x)M_y(x)]_a^b - \int_I \frac{d(\delta \varphi_y(x))}{dx} M_y(x) \, dx - \int_I \delta \varphi_y(x)Q_z(x) \, dx
\]

• Enforcement of boundary conditions

\[
0 = [\delta w(x)\overline{Q}_z(x)]_{I_p} - \int_I \frac{d(\delta w(x))}{dx} \overline{Q}_z(x) \, dx + \int_I \delta w(x) \overline{f}_z(x) \, dx
\]

\[
0 = [\delta \varphi_y(x)\overline{M}_y(x)]_{I_p} - \int_I \frac{d(\delta \varphi_y(x))}{dx} \overline{M}_y(x) \, dx - \int_I \delta \varphi_y(x)\overline{Q}_z(x) \, dx
\]
• The weak of equilibrium equations (we insert \(1\) for \(M_y\) and \(2\) for \(Q_z\))

\[
\int_I \frac{d(\delta w(x))}{dx} k(x) G(x) A(x) \left( \frac{dw(x)}{dx} + \varphi_y(x) \right) \, dx = \\
\left[ \delta w(x) \overline{Q_z}(x) \right]_{I_p} + \int_I \delta w(x) \overline{f_z}(x) \, dx
\]  
(7)

\[
\int_I \frac{d(\delta \varphi_y(x))}{dx} E(x) I_y(x) \frac{d\varphi_y(x)}{dx} \, dx + \\
\int_I \delta \varphi_y(x) k(x) G(x) A(x) \left( \frac{dw(x)}{dx} + \varphi_y(x) \right) \, dx = \\
\left[ \delta \varphi_y(x) \overline{M_y}(x) \right]_{I_p}
\]  
(8)
7 FEM discretization

- We replace a continuous structure with $n$ nodal points and $(n - 1)$ (finite) elements.
- In every nodal point we introduce two independent quantities – a deflection $w_i$ and a rotation $\varphi_{yi}$ of the $i$-th nodal point.
- On the level of whole structure, we collect the unknowns into vectors of deflections $r_w$ and rotations $r_\varphi$.
- Discretization of unknown quantities and their derivatives

\[
\begin{align*}
  w(x) & \approx N_w(x)r_w, \\
  \frac{d}{dx}w(x) & \approx B_w(x)r_w, \\
  \varphi_y(x) & \approx N_\varphi(x)r_\varphi, \\
  \frac{d}{dx}\varphi_y(x) & \approx B_\varphi(x)r_\varphi.
\end{align*}
\]
• Discretization of weight functions

\[ \delta w(x) \approx N_w(x) \delta r_w \quad \frac{d(\delta w(x))}{dx} \approx B_w(x) \delta r_w \]

\[ \delta \varphi_y(x) \approx N_\varphi(x) \delta r_\varphi \quad \frac{d(\delta \varphi_y(x))}{dx} \approx B_\varphi(x) \delta r_\varphi \]

• The linear system of discretized equilibrium equations

\[
\begin{bmatrix}
K_{ww} & K_{w\varphi} \\
K_{\varphi w} & K_{\varphi\varphi}
\end{bmatrix}
\begin{bmatrix}
r_w \\
r_\varphi
\end{bmatrix}
= 
\begin{bmatrix}
R_w \\
R_\varphi
\end{bmatrix}
\]

• Compact notation

\[
K_{(2n \times 2n)} r_{(2n \times 1)} = R_{(2n \times 1)}
\]
8 Shear locking

- \( K_{\varphi w} = K_{w\varphi}^T \) \( \Rightarrow \) the stiffness matrix \( K \) is symmetric thanks to appearance of the terms \( \int_I (\delta w(x))'kGA(x)\varphi_y(x) \, dx \) in (7) and \( \int \delta \varphi_y(x)kGA(x)w'(x) \, dx \) in (8).

**Homework 2.** Derive explicit relations for matrices \( K_{ww}, K_{w\varphi}, K_{\varphi w}, K_{\varphi\varphi} \) and vectors \( R_w, R_\varphi \).

8 Shear locking

- For \( h/L \to 0 \), the response of a Mindlin theory-based element should approach the classical slender beam (negligible shear effects).
- If the basis functions \( N_w \) a \( N_\varphi \) are chosen as piecewise linear, resulting response in too “stiff” \( \to \) excessive influence of shear terms, sc. shear locking.
8.1 Statics-based analysis

- Shear force: \( Q_z(x) = k(x)G(x)A(x) \left( \frac{dw(x)}{dx} + \varphi_y(x) \right) \) – linear

- Bending moment: \( M_y(x) = E(x)I_y(x)\frac{d\varphi_y(x)}{dx} \) – constant

- Severe violation of the Schwedler relation

\[
\frac{dM_y(x)}{dx} - Q_z(x) = 0
\]

8.2 Kinematics-based explanation
The approximate solution must be able to correctly reproduce the pure bending mode, see [3, Section 3.1]):

\[ \kappa_y(x) = \frac{d\varphi_y(x)}{dx} = \kappa = \text{const} \]

\[ \gamma_{zx}(x) = \frac{dw(x)}{dx} + \varphi_y(x) = 0 \]

For the given discretization

\[ w(x) \approx w_1 \left(1 - \frac{x}{L}\right) + w_2 \frac{x}{L} \]

\[ \frac{dw(x)}{dx} \approx \frac{1}{L} (w_2 - w_1) \]

\[ \varphi_y(x) \approx \varphi_1 \left(1 - \frac{x}{L}\right) + \varphi_2 \frac{x}{L} \]

\[ \frac{d\varphi_y(x)}{dx} \approx \frac{1}{L} (\varphi_2 - \varphi_1) \]

The requirement of zero shear strain leads to

\[ \gamma_{zx}(x) \approx \frac{1}{L} (w_2 - w_1) + \varphi_1 + \frac{x}{L} (\varphi_2 - \varphi_1) = 0. \]

Therefore, the previous relation must be independent of the \(x\) coordinate \(\Rightarrow\)

\[ \varphi_2 - \varphi_1 = 0 \Rightarrow \kappa_y \approx \frac{1}{L} (\varphi_2 - \varphi_1) = 0 \neq \kappa \]
9 Selective integration

- The shear strain is assumed to be constant on a given interval, its value is derived from the value in the center of an interval

\[ \gamma_{zx}(x) \approx \frac{1}{L}(w_2-w_1)+\varphi_1+\frac{1}{2}(\varphi_2-\varphi_1) = \frac{1}{L}(w_2-w_1)+\frac{1}{2}(\varphi_1+\varphi_2) \]

- Kinematics: the element behaves correctly, it enables to describe the pure bending mode.

- Statics: \( Q_z(x) = k(x)G(x)A(x)\gamma_{xz}(x) - \text{constant}, \ M_y - \text{constant} \) ← the Schwedler condition is not "severely violated".

10 Bubble (hierarchical) function

- It follows from analysis of the kinematics that the shear locking is caused by insufficient degree of polynomial approximation of the dis-
placement \( w(x) \).

- Therefore, we add a quadratic term to approximation of \( w(x) \):

  \[
  w(x) \approx w_1 \left( 1 - \frac{x}{L} \right) + w_2 \frac{x}{L} + \alpha x(x - L)
  \]

- Pure bending mode requirement

  \[
  \gamma_{zx}(x) = \frac{dw(x)}{dx} + \varphi_y(x)
  \]

  \[
  \approx \frac{1}{L} (w_2 - w_1) + \alpha (2x - L) + \varphi_1 + \frac{x}{L} (\varphi_2 - \varphi_1)
  \]

  \[
  = \frac{1}{L} (w_2 - w_1) - \alpha L + \varphi_1 + \frac{x}{L} (\varphi_2 - \varphi_1 + 2\alpha L)
  \]

  \[
  = 0
  \]
• Requirement of independence of coordinate $x \Rightarrow$

\[
\alpha = \frac{1}{2L} (\varphi_1 - \varphi_2)
\]

• Final approximations

\[
\begin{align*}
w(x) & \approx w_1 \left( 1 - \frac{x}{L} \right) + w_2 \frac{x}{L} + \frac{1}{2L} (\varphi_1 - \varphi_2) x(x - L) \\
\varphi_y(x) & \approx \varphi_1 \left( 1 - \frac{x}{L} \right) + \varphi_2 \frac{x}{L}
\end{align*}
\]

• From the “static” point of view the element behaves similarly to previous formulation – $Q_z$ is constant, $M_y$ is constant.

• Approximation of the $w$ displacement not based not only on the values of deflections nodal, but also on the values of nodal rotations \[2\] – sc. linked interpolation.
Method of Lagrange multipliers

Recall the weak form of the bending moment equilibrium equations (8) for a beam with $M_y = 0$, constant values of $E$, $G$ and a rectangular cross-section.

$$
0 = EI_y \int_I \frac{d(\delta \varphi_y(x))}{dx} \frac{d\varphi_y(x)}{dx} \, dx + kGA \int_I \delta \varphi_y(x) \left( \frac{dw(x)}{dx} + \varphi_y(x) \right) \, dx
$$

$$
= E \frac{bh^3}{12} \int_I \frac{d(\delta \varphi_y(x))}{dx} \frac{d\varphi_y(x)}{dx} \, dx
$$

$$
+ \frac{5}{6} \frac{E}{2(1 + \nu)} bh \int_I \delta \varphi_y(x) \left( \frac{dw(x)}{dx} + \varphi_y(x) \right) \, dx \left/ \frac{12}{Ebh^3} \right.
$$

$$
\int_I \frac{d(\delta \varphi_y(x))}{dx} \frac{d\varphi_y(x)}{dx} \, dx + \frac{5}{1 + \nu} \frac{1}{h^2} \int_I \delta \varphi_y(x) \left( \frac{dw(x)}{dx} + \varphi_y(x) \right) \, dx = 0
$$

The condition of zero shear strain for $h \to 0$ is imposed via the sc. penalty term.
• For slender beams and linear-linear approximation this leads to the shear locking as

\[
\lim_{h \to 0} \int_I \frac{1}{h^2} \delta \varphi_y(x) \left( \frac{d w(x)}{dx} + \varphi_y(x) \right) \, dx = 0.
\]

• If we introduce a new independent variable for imposing the condition \( \gamma_{xz} = 0 \) for \( h \to 0 \), we suppress influence of the choice of approximation of unknowns \( w(x) \) a \( \varphi_y(x) \).

• Therefore, we have to add an additional condition to weak equilibrium equations (7)–(8)

\[
\int_I \delta \lambda(x) \left( \gamma_{zx}(x) - \frac{d w(x)}{dx} - \varphi_y(x) \right) \, dx = 0, \quad (9)
\]

where \( \gamma(x) \) is now a new variable independent of \( w \) and \( \varphi_y \) and \( \delta \lambda(x) \) is the corresponding weight function.
• Constitutive equations for the shear force $Q_z$ now simplify as

$$Q_z(x) = k(x)G(x)A(x)\gamma_{xz}(x).$$

• Weak form of equilibrium of equations can now be rewritten as

$$0 = \int_I \frac{d(\delta w(x))}{dx} k(x)G(x)A(x)\gamma_{xz}(x) \, dx - \left[\delta w(x)\overline{Q_z(x)}\right]_{Ip}$$

$$- \int_I \delta w(x) f_z(x) \, dx$$

$$0 = \int_I \frac{d(\delta \varphi_y(x))}{dx} E(x)I_y(x) \frac{d\varphi_y(x)}{dx} \, dx$$

$$+ \int_I \delta \varphi_y(x) k(x)G(x)A(x)\gamma_{xz}(x) \, dx - \left[\delta \varphi_y(x)\overline{M_y(x)}\right]_{Ip}$$

$$0 = \int_I \delta \lambda(x) \left(\gamma_{xz}(x) - \frac{dw(x)}{dx} - \varphi_y(x)\right) \, dx$$

• Observe that is we choose the weight function in the specific form

$$\delta \lambda(x) = k(x)G(x)A(x)\delta \gamma_{xz}(x),$$
we will finally obtain a symmetric stiffness matrix $K$.

- The last equation now can be modified as

$$0 = \int_I \delta \gamma_{xz}(x) k(x) G(x) A(x) \left( \gamma_{zx}(x) - \frac{dw(x)}{dx} - \varphi_y(x) \right) \, dx.$$ 

- The additional variable $\gamma_{xz}$ needs to be discretized

$$\gamma_{xz}(x) \approx \underbrace{N_\gamma(x)} r_\gamma$$

and inserted into the weak form of equilibrium equations. This yields, after standard manipulations, the following system of linear equations

$$\begin{bmatrix}
K_{ww} & K_{wp} & K_{w\gamma} \\
K_{p\varphi} & K_{\varphi\varphi} & K_{\varphi\gamma} \\
K_{\gamma w} & K_{\gamma p} & K_{\gamma\gamma}
\end{bmatrix}
\begin{bmatrix}
r_w \\
r_\varphi \\
r_\gamma
\end{bmatrix}
= 
\begin{bmatrix}
R_w \\
R_{\varphi} \\
0
\end{bmatrix}$$

- The stiffness matrix, resulting from this discretization, is larger only virtually. It can be observed that parameters $r_\gamma$ only internal and can
be eliminated (expressed via variables \( r_w \) and \( r_\varphi \)); see, e.g. [1, pp. 234–235] for more details.

- This formulation works even for piecewise linear approximation of \( w \) and \( \varphi_y \); it suffices to approximate \( \gamma \) as a piecewise constant on an element.

- Kinematics: shear locking avoided due to (9).

- Statics: the shear force \( Q_z \) is again (piecewise) constant, so is the bending moment \( M_y \).

**Homework 3**. Derive the element stiffness matrix based on Lagrange multipliers. Assume the linear approximation of deflections \( w(x) \), linear approximation of rotations \( \varphi_y(x) \) and constant values of \( \gamma_{xz} \) on a given elements. Show that this procedure yields results identical to the reduced integration and linked interpolation.
A humble plea. Please feel free to e-mail any suggestions, errors and typos to zemanj@cml.fsv.cvut.cz.

References


